The Nature of the Dirac Equation
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Introduction

The Dirac Equation[1] is a staple of relativistic quantum theory and is widely applied to objects such as electrons and protons.

$$i\hbar\frac{\partial\mathbf{\Psi}}{\partial t}=\frac{c\hbar}{i}\sum_n \boldsymbol{\alpha}_n\cdot\left(\frac{\partial\mathbf{\Psi}}{\partial x_n}\right)+m_0c^2\boldsymbol{\beta}\cdot\mathbf{\Psi} \tag{1}$$

The $\boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}$ are 4×4 matrices, the $\mathbf{\Psi}$ is a 4-component column matrix and m_0 is the rest mass. The motivation behind this piece is to explore this equation, what does it say and mean.

Reworking the equation

It can be shown that the solutions to (1) must also satisfy the Klein-Gordon equation[2 3], which is a scalar equation.

$$\nabla^2\Psi-\frac{1}{c^2}\cdot\frac{\partial^2\Psi}{\partial t^2}=\left(\frac{m_0c}{\hbar}\right)^2\Psi \tag{2}$$

In this light the basis set to the solution of (1) can be expressed as the product of a scalar solution to the Klein-Gordon equation, call it Ψ_k, along with a column matrix consisting of constants, call it **K**.

$$\mathbf{\Psi}=\boldsymbol{K}\Psi_k \tag{3}$$

Putting (3) into (1) and utilizing the assumption that **K** is constant.

$$i\hbar\frac{\partial(\boldsymbol{K}\Psi_k)}{\partial t}=\frac{c\hbar}{i}\sum_n \boldsymbol{\alpha}_n\cdot\left(\frac{\partial(\boldsymbol{K}\Psi_k)}{\partial x_n}\right)+m_0c^2\boldsymbol{\beta}\cdot(\boldsymbol{K}\Psi_k)$$

$$\boldsymbol{K}\cdot i\hbar\frac{\partial\Psi_k}{\partial t}=\frac{c\hbar}{i}\sum_n \boldsymbol{\alpha}_n\cdot\boldsymbol{K}\left(\frac{\partial\Psi_k}{\partial x_n}\right)+m_0c^2\boldsymbol{\beta}\cdot\boldsymbol{K}\Psi_k \tag{4}$$

We can reduce this down to a scalar equation by performing a matrix multiplication with the transpose of the complex conjugate of **K**. This is to allow for potentially complex elements.

$$\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{K}\cdot i\hbar\frac{\partial\Psi_k}{\partial t}=\frac{c\hbar}{i}\sum_n \bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{\alpha}_n\cdot\boldsymbol{K}\left(\frac{\partial\Psi_k}{\partial x_n}\right)+m_0c^2\,\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{\beta}\cdot\boldsymbol{K}\Psi_k$$

$$i\hbar\frac{\partial\Psi_k}{\partial t}=\sum_n\left(\frac{c}{|K|^2}\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{\alpha}_n\cdot\boldsymbol{K}\right)\cdot\left(\frac{\hbar}{i}\cdot\frac{\partial\Psi_k}{\partial x_n}\right)+m_0c^2\left(\frac{1}{|K|^2}\cdot\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{\beta}\cdot\boldsymbol{K}\right)\Psi_k \tag{5a}$$

with $\quad |K|^2\equiv\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{K}$ (5b)

The portions of (5a) contained in the parentheses are in fact scalars. With the appropriate substitutions this can be re-written as:

$$i\hbar\frac{\partial\Psi_k}{\partial t}=\vec{V}\cdot\left(\frac{\hbar}{i}\cdot\vec{\nabla}\Psi_k\right)+m_0c^2\,\Gamma\,\Psi_k \tag{6a}$$

$$V_n=\frac{c}{|K|^2}\cdot\left(\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{\alpha}_n\cdot\boldsymbol{K}\right) \tag{6b}$$

$$\Gamma=\frac{1}{|K|^2}\cdot\left(\bar{\boldsymbol{K}}^{\boldsymbol{T}}\cdot\boldsymbol{\beta}\cdot\boldsymbol{K}\right) \tag{6c}$$

At this point the matrix equation (1) has been transformed into scalar equation (6a). As stated above Ψ_k must be a solution to the Klein-Gordon equation, therefore equation (6a) can be viewed as a boundary condition to be satisfied on (2). But what does **K** mean? Thus far there are two things that can be deduced. First, it is needed to interface the scalar Ψ_k into (1). Second, it influences the boundary condition (6a). Yet there is still more to be learned.

Connecting to relativity

Before exploring **K** some non-quantum mechanical physics needs to be explored. The mass-shell relationship is[4]

$$E^2=c^2p^2+(m_0c^2)^2 \tag{7}$$

This can be linearized using the standard equations for relativistic energy and momentum[5].

$$\left(m_0\gamma c^2\right)E=c^2\left(m_0\gamma\vec{u}\right)\cdot\vec{p}+m_0^2c^4$$

$$E=\vec{u}\cdot\vec{p}+m_0c^2\cdot\frac{1}{\gamma} \tag{8}$$

For energy, momentum eigenstates states (6a) and (8) are compatible with two assumptions

$$\vec{V}\rightarrow\vec{u}\qquad\qquad\Gamma\rightarrow\frac{1}{\gamma} \tag{9}$$

The nature of the solution

So what is the significance of the elements of **K**? To determine this substitute the standard expressions for α_n and β into (6a) and (6b). This gives

$$V_1 = \frac{2c}{|K|^2} \cdot \left(\Re(\bar{K}_1 K_4) + \Re(\bar{K}_2 K_3) \right) \tag{10a}$$

$$V_2 = \frac{2c}{|K|^2} \cdot \left(\Re(\bar{K}_1 K_3) - \Re(\bar{K}_2 K_4) \right) \tag{10b}$$

$$V_3 = \frac{2c}{|K|^2} \cdot \left(\Im(\bar{K}_1 K_4) - \Im(\bar{K}_2 K_3) \right) \tag{10c}$$

$$\Gamma = \frac{|K_1|^2 + |K_2|^2 - |K_3|^2 - |K_4|^2}{|K|^2} = \frac{1}{\gamma} \tag{10d}$$

The conclusions are interesting.

- Because of (10d), states where K_1 and K_2 are the only nonzero elements represent positive energy while states where K_1 and K_2 are the only nonzero elements represents negative energy.

- An examination of equations (10a) to (10c) shows that states with only one non-zero element of **K** will represent a state at rest.

- In order for there to be motion one needs to have either K_1 or K_2 as a non-zero elements and either K_3 or K_4 as a non-zero elements.

Because motion in a particular direction, ala (10), does not depend on a single element of **K**, it cannot represent a vector, at least not in space-time.

The state of the Union

An interesting consequence arises from the third bullet point above. Consider three scenarios below.

K matrix	Γ	Energies
$\boldsymbol{K}=\begin{pmatrix} K_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	1	$E = m_0c^2$
$\boldsymbol{K}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ K_4 \end{pmatrix}$	-1	$E = -m_0c^2$
$\boldsymbol{K}=\begin{pmatrix} K_1 \\ 0 \\ 0 \\ K_4 \end{pmatrix}$	$-1\leq\Gamma\leq1$	$\lvert E\rvert \geq m_0c^2$

Table 1.

A state with both K_1 and K_4 elements will be in motion and so will have a positive energy greater than its rest energy or a negative energy less than its negative rest energy.

A conceptual conflict arises if we envision each element of **K** as a separate state. The reason for the conflict is that if K_1 and K_4 represent states then the third state in the above table should constitute a mixture of positive and energy states and so having an energy

$$|E|\leq m_0 c^2 \tag{11}$$

should be possible. However it is not. It is therefore logically inconsistent to assign a state to each element. All that one can say is that a particular configuration of **K** leads to a particular state.

Spin

In the literature the first and third elements of **Ψ** are assigned to the spin up states while the balance belong to spin down. However, as can be seen from the above arguments, this is a poor interpretation. The most that can be said is that a state with the only nonzero elements being K_1 and K_3 can be assigned the label of "spin-up." Likewise states with only nonzero elements being K_2 and K_4 can be called "spin-down." However simply classifying other states as a mixture of spin-up and spin down is problematic.

The difficulty lies in not being able to separate any **Ψ** into spin-up and spin-down states.

The reason is that while **K** can be so divided the associated $\mathbf{\Psi}_k$ from (3) cannot, for when one divides K the associated $\mathbf{\Psi}_k$ would have to change

So what can be said about spin according to Dirac? According to the paradigm, the Dirac equation shows spin as an intrinsic property[6]. However the derivation itself does not include any information about any of the intrinsic properties of the particle in question. Moreover this solution is said to be the solution for matter with two spin states, called spin ½ particles, while the Klein-Gordon equation is only valid for particles without spin. Yet as was seen above the elements of $\mathbf{\Psi}$ cannot be assigned to specific states. Thus the question of what (1) says about spin is still left open and vaque.

Connecting the Dirac and Klein-Gordon equations via (8) shows the underlying physics behind the derivations. Indeed both are extensions of (7). So whereas the Klein-Gordon is a second order scalar equation the Dirac is a first order matrix equation. Neither equation contains neither extra information nor a deficit of information and so the two tell the same story, thus demonstrating a weakness in the orthodox interpretation of the elements of $\mathbf{\Psi}$.

Works Cited

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